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# On the Initial Boundary Value Problems for the Enskog Equation in Irregular Domains

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The paper is concerned with the Enskog equation with a constant high density factor for large initial data in $L^1(R^n)$. The initial boundary value problem is investigated for bounded domains with irregular boundaries. The proof of an H-theorem for the case of general domains and boundary conditions is given. The main result guarantees the existence of global solutions of boundary value problems for large initial data with all $v$-moments initially finite and domains having boundary with finite Hausdorff measure and satisfying a cone condition. Existence and uniqueness are first proved for the case of bounded velocities. The solution has finite norm $\int (\sup_{0 \leqslant t \leqslant T} f(t_0 + t, x + vt, v)) \sqrt{1 + v^2}\, dq\, dv$, where $q = (t_0, x)$ is taken on all possible $n$-dimensional planes $Q(v)$ in $R^{n+1}$ intersecting a fixed point and orthogonal to vectors $(1, v)$, $v \in R^n$.

**KEY WORDS:** Enskog equation; irregular domains; H-theorem; initial boundary value problem.

## 1. INTRODUCTION

The Enskog equation is a popular model of transport processes in moderately dense gases.[10] Two alternative approaches based on different ideas are possible for the mathematical investigation of this equation. An important specific regularity property of the Enskog equation was observed by Arkeryd.[1] This property gave a global existence results for large initial data in dimensions one and two. Similar ideas were used in ref. 2 for getting existence results for the modified Boltzmann equation.

Getting global results for the Enskog equation with large initial data in the style of Di Perna and Lions or more regular solutions faces an obstacle, in that classical entropy estimates for the Enskog equation do not

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exist. The lack of symmetry in the Enskog equation implies that the classical sign estimate for the entropy production is no longer valid.

Resibois in ref. 15 introduced a new modified $H$-functional for the Enskog equation. Estimates for this functional useful for an existence proof were given in ref. 4.

This modified $H$-functional and the appropriate entropy estimates for the case of an infinite space and for a periodical box was an essential ingredient in the theory of global existence in $L^1$,[4] and for the global existence and uniqueness in the class of functions with the norm $\int (\sup_{0 \leqslant t \leqslant T} f(t, x+vt, v))(1+v^r)\, dx\, dv$.[3]

In the present paper we demonstrate an approach for the well posedness of the initial boundary value problems for the Enskog equation in a bounded domain with irregular boundary and with boundary conditions of general diffuse reflection form.

In the case of smooth domains useful estimates for the traces of solutions of the kinetic equations which preserve mass can be proved from some information about the solution inside the domain.[6] In irregular domains this approach does not work, and more detailed analysis of the solutions close to the boundary is necessary. A general analysis of such problems for kinetic equations was given in refs. 12–14. Results from ref. 14 are essentially used in the present paper. We use the representation $f = f_i + f_e$ for the solution $f$, where $f_i$ is a solution of the collisionless problem for bounded approximate initial data having compact support. A new estimate is proved for the entropy flux for the collisionless problem and irregular boundaries. This estimate is important for the limits of the approximate solutions of the boundary value problems for nonlinear kinetic equations. Here finiteness of the Hausdorff measure of the boundary, a kind of cone condition, and a regularity property for the reflection operator on the boundary are assumed.

The non-local nature of the Enskog collision term causes additional problems for estimates of the entropy production in the case of domains with nontrivial boundary conditions. One of the results of the present paper is an H-theorem for the Enskog equation in irregular domains with the boundary condition of general diffuse reflection form.

The main results of the paper are existence and uniqueness of solutions for the initial boundary value problem of the Enskog equation in irregular domains and large $L^1$ initial data with all $v$-moments initially finite. This is first proved for bounded and then for unbounded velocities. The solutions have finite norms which are supremum over collisionless trajectories $(t, x+vt)$ from the boundary of the domain of $(t, x)$ variables averaged with weights $(1+v^r)$ over these trajectories and over the ingoing velocities $v$. Our analysis is a combination of methods from the papers.[3, 14]

The boundary conditions introduced here are a straightforward analogy of the boundary conditions for the Boltzmann equation. The case of the initial boundary value problems for the Boltzmann equation and for the Enskog equation in irregular domains with solutions in $L^1$ will be presented in a forthcoming paper.

## 2. BASIC EQUATIONS AND THE ENTROPY INEQUALITIES

The representation of the Enskog collision operator in the whole of $R^n$ or the periodic case is as follows:

$$Q(f,f) = \int_{\mathscr{L}_+ \times R^n} (f'f'_-k_- - ff_+k_+)\, S(v, v_*, u)\, dv_*\, du \tag{2.1}$$

$$S \equiv S(v, v_*, u) = \sigma^2 \max((v - v_*) \cdot u, 0) \tag{2.2}$$

where $u$ varies on the hemisphere $\mathscr{L}_+ = (u\colon |u| = 1, (v - v_*) \cdot u \geqslant 0)$, $k_-$, $k_+$ are functions of the local density at the point $(x - \sigma u)$ and at the point $(x + \sigma u)$.

The arguments in $f'$, $f'_- = f'_*$, $f$, $f_+ = f_*$ are: $(x, v')$, $(x - \sigma u, v'_*)$, $(x, v)$, $(x + \sigma u, v_*)$, where

$$v' = v - u((v - v_*) \cdot u), \qquad v'_* = v_* + u((v - v_*) \cdot u) \tag{2.3}$$

$v$ and $v_*$ are pre-collisional velocities of two molecules, $v'$ and $v'_*$ are their velocities after collision. In the following we assume the density factors constant, $k_- = k_+ = 1$.

When the gas is situated in some domain $\Omega$ with boundary $\partial\Omega$ it is natural to assume the functions under the integral in (2.1) to be zero when the space argument is outside $\Omega$.

Let $\mathscr{L}_+(\Omega) = \{u \in \mathscr{L}_+ : x + u\sigma \in \Omega\}$, $\mathscr{L}_-(\Omega) = \{u \in \mathscr{L}_+ : x - u\sigma \in \Omega\}$ and $\chi_-$, $\chi_+$, and $\chi$ be the characteristic functions of $\mathscr{L}_+(\Omega)$, $\mathscr{L}_-(\Omega)$, and $\Omega$. Then our assumption implies the following representation for the collision operator:

$$Q(f,f) = \sigma^2 \int_{\mathscr{L}_+ \times R^n} (f'f'_-\chi\chi_- - ff_+\chi\chi_+)((v - v_*) \cdot u)\, dv_*\, du \tag{2.4}$$

To simplify notations and to exclude characteristic functions from formulas we assume in the following that all functions of the space variable are zero outside of $\Omega$ and that actual domains of integration are delimited by this requirement.

The Enskog equation depending on the variables $(t, x, v) \in R_+ \times \Omega \times R^n$ is

$$\frac{\partial f}{\partial t} + v\nabla_x f = Q(f, f) \tag{2.5}$$

We assume the boundary conditions at the boundary $\partial\Omega$ of the domain $\Omega$ to be of the same type as for the Boltzmann equation. Let $f^+$ and $f^-$ be the distribution functions of molecules, falling on to and reflected by $\partial\Omega$. Then

$$f^+ = Rf^-, \qquad x \in \partial\Omega \tag{2.6}$$

$$f^-(t, x, v) = \begin{cases} f(t, x, v), & \text{if} \quad x \in \partial\Omega, \quad v \cdot n(x) < 0 \\ 0, & \text{if} \quad x \in \partial\Omega, \quad v \cdot n(x) \geqslant 0 \end{cases}$$

$$f^+(t, x, v) = f(t, x, v) - f^-(t, x, v), \qquad \text{if} \quad x \in \partial\Omega$$

where $n(x)$ is a unit normal to $\partial\Omega$, at the point $x$ directed towards the interior of $\Omega$. The operator $R$ is assumed to be linear and of local type. At each $x \in \partial\Omega$ and at each time $t$ it is defined as an operator, acting on functions of the argument $v$ alone, with the parametric dependence of $x$. The case of irregular boundary $\partial\Omega$ will be discussed in the second section.

If there is no emission and absorption of molecules on $\partial\Omega$, then the operator $R$ satisfies the following condition:

$$\int_{R^n} Rf^- \,|v \cdot n|\, dv = \int_{R^n} f^- \,|v \cdot n|\, dv \tag{2.7}$$

The last relation represents a local mass conservation law during reflection of molecules on $\partial\Omega$.

Usually it is required that a gas with Maxwell distribution

$$M(x, v) = \frac{\theta^2(x)}{2\pi} \exp\left\{\frac{1}{2\theta(x)} |v|^2\right\} \tag{2.8}$$

interacting with a wall, having the same temperature $\theta(x)$, must be in a local equilibrium with the wall. This property implies the assumption:

$$M^+ = RM^- \tag{2.9}$$

for the operator $R$. As a consequence, if the wall $\partial\Omega$ is motionless and has a constant temperature, then the Maxwell distribution, with corresponding constant parameters is an exact stationary solution of the Eq. (2.5) with the boundary conditions (2.6).

We will prove a modified form of the $H$-theorem valid for the Enskog equation in the case of the gas contained in the domain $\Omega$ with the boundary condition (2.6) satisfying the property (2.7).

The symmetry properties of the Enskog collision operator are valid also for the form (2.4) of the collision operator in the domain $\Omega$ and imply the usual relation:

$$\int_{R^n\times\Omega} Q(f,f)\,\varphi(x,v)\,dv\,dx = \tfrac{1}{2}\sigma^2 \int_{\mathscr{S}_+\times R^n\times R^n\times\Omega} (\varphi' + \varphi'_- - \varphi - \varphi_+) \times ff_+((v-v_*)\cdot u\,)\,dv_*\,du\,dx\,dv \tag{2.10}$$

which implies mass and energy conservation properties.

Several of the following steps of our proof use arguments from ref. 4 for the case without boundary. They are still valid because one only uses simple changes of variables like reflection with respect to $u$ and shifts with respect to $x$, keeping the estimated integrals invariant independently of the shape of $\Omega$. In contrast to ref. 4 we estimate the entropy production during some interval of time. This has some advantages in the case of bounded domains.

Using (2.10) with $\varphi(x,v)=\ln(f)$, the invariance of the integral over $dx$ under shifts, the relation $dv\,dv_* = dv'\,dv'_*$, and the classical inequality $g(\log g - \log h)\geqslant g-h$, we get the following estimate for the entropy production term:

$$\int_0^T \int_{R^n\times\Omega} Q(f,f)\log f\,dv\,dx\,dt \leqslant \Gamma \tag{2.11}$$

where

$$\Gamma = \tfrac{1}{2}\sigma^2 \int_0^T \int_{\mathscr{S}_+\times R^n\times R^n\times\Omega} (ff_- - ff_+)(v-v_*)\cdot u\,dv_*\,du\,dv\,dx\,dt \tag{2.12}$$

Let us introduce the following notations:

$$\rho_\pm = \int f_\pm\,dv; \qquad j_\pm = \int vf_\pm\,dv \tag{2.13}$$

The symmetry properties of the terms in $\Gamma$ when changing the magnitude of the vector $u$, and the invariance of involved integrals under shifts for $\sigma u$ and $-\sigma u$ in the space variables, give that

$$\Gamma = \tfrac{1}{2}\sigma^2 \int_0^T \int_{\mathscr{S}\times\Omega} (j\rho_- - j_-\rho)\cdot u\,du\,dx\,dt \tag{2.14}$$

Several suitable changes of the space variables in the terms under the last integral sign, imply the following relation:

$$\Gamma = \sigma^2 \int_0^T \int_{\mathscr{L}(\Omega)\times\Omega} \rho j_+ \cdot u \, du \, dx \, dt$$

$$= \sigma^2 \int_0^T \int_{\mathscr{L}(\Omega)\times\Omega} \rho(x) \, j(x+\sigma u) \cdot u \, du \, dx \, dt \tag{2.15}$$

where $\mathscr{L}(\Omega) = \{u \in \mathscr{L}: x + \sigma u \in \Omega\}$.

At this point we use the property (2.7) of the boundary conditions. It implies that

$$\int f(x, v) \, v \cdot n(x) \, dv = j(x) \cdot n(x) = 0 \qquad \text{for} \quad x \in \partial\Omega \tag{2.16}$$

Let us introduce notations: $\mathscr{B}_\sigma = \{y: |x-y| \leqslant \sigma\}$, $\mathscr{B}_\sigma(\Omega) = \mathscr{B}_\sigma \cap \Omega$ for the intersection of the ball $\mathscr{B}_\sigma$ with the domain $\Omega$ and $\partial\mathscr{B}_\sigma(\Omega) = \mathscr{L}(\Omega) \cup (\mathscr{B}_\sigma \cap \partial\Omega)$ for the boundary of $\mathscr{B}_\sigma(\Omega)$. Then

$$\Gamma = \sigma^2 \int_0^T \int_{\partial\mathscr{B}_\sigma(\Omega)\times\Omega} \rho(x) \, j(y) \cdot n(y) \, dS_y \, dx \, dt$$

Using the mass conservation for the Enskog equation in $[0, T] \times \mathscr{B}_\sigma(\Omega) \times R^n$, we obtain:

$$\Gamma = \sigma^2 \int_0^T dt \int_\Omega dx \, \rho(x, t) \frac{\partial}{\partial t} \int_0^t ds \int_{\partial\mathscr{B}_\sigma} j(y, s) \cdot n(y) \, dS_y$$

$$= \sigma^2 \int_\Omega dx \int_0^T dt \, \rho(x, t) \frac{\partial}{\partial t} (m(\mathscr{B}_\sigma(\Omega), 0) - m(\mathscr{B}_\sigma(\Omega), t)) \tag{2.17}$$

where $m(\mathscr{B}_\sigma(\Omega), t)$ is a mass of the gas in $\mathscr{B}_\sigma(\Omega)$ at time $t$:

$$m(\mathscr{B}_\sigma(\Omega), t) = \int_{\mathscr{B}_\sigma(\Omega)} \rho(y, t) \, dy \tag{2.18}$$

Integration by parts over $t$ in (2.17) gives the following result.

**Lemma 2.1.** If the boundary condition for the Enskog equation has the mass conservation property (2.7), then the following estimate for the entropy production term is valid:

$$\int_0^t \int_\Omega \int_{R^n} Q(f, f) \ln f \, dx \, dv \, ds$$
$$\leqslant \tfrac{1}{2}\sigma^2 \left( \int_\Omega \rho(x, 0)\, m(\mathscr{B}_\sigma(\Omega), 0)\, dx - \int_\Omega \rho(x, t)\, m(\mathscr{B}_\sigma(\Omega), t)\, dx \right) \tag{2.19}$$

In the case of constant temperature on the boundary, the Darrozes–Guiraud inequality, Lemma 2.1, estimates for energy and entropy given in ref. 3 for the case of the Boltzmann equation, and an argument from ref. 7 imply the following variant of the $H$-theorem for the Enskog equation.

**Theorem 2.2.** Let the temperature on $\partial\Omega$ be constant, let $\partial\Omega$ have $H^{n-1}$ finite Hausdorff measure. If then boundary conditions preserve mass and the initial data $F_0$ satisfy conditions $F_0(1 + v^2 + \ln F_0) \in L^1(\Omega \times R^n)$, then the functional

$$H(f) = \int_{\Omega \times R^n} f \ln f \, dx \, dv + \frac{\sigma^2}{2} \int_{\mathscr{B}_\sigma(\Omega) \times \Omega} \rho(x, t)\, \rho(y, t)\, dy \, dx \tag{2.20}$$

is a decreasing function of $t$, and the following estimates for solutions of the Enskog equation are valid for arbitrary $t$:

$$\int_{\Omega \times R^n} f(t, x, v)(1 + v^2)\, dx \, dv < C \tag{2.21}$$

$$\int_{\Omega \times R^n} f(t, x, v) \ln f \, dx \, dv < \int_{\Omega \times R^n} F_0(t, x, v) \ln F_0 \, dx \, dv + \frac{\sigma^2}{2} \left( \int_{\Omega \times R^n} F_0 \, dx \, dv \right)^2 \tag{2.22}$$

$$\int_{\Omega \times R^n;\, f > w} f(t, x, v)\, dx \, dv < \frac{1}{\ln w}\, C \tag{2.23}$$

where constants $C$ are dependent on $F_0$ and independent of $t$.

The strict proof of the lemma and the theorem in full generality for irregular domains follows from Lemma 3.4, Proposition 5.25 and approximation.

Estimates (2.21)–(2.23) imply that for a fixed large interval of time, $w < 2^{j-1}$ can be chosen such that

$$\int_{\substack{\Omega \times R^n \\ |v| > w \text{ or } F_0 > w}} \max(1, |v|)\, F_0\, dx\, dv < (128)^{-1}\, C(\Omega, R) \tag{2.24}$$

where below the constant $C(\Omega, R)$ will be chosen depending on the solution of the collisionless problem in the domain $\Omega$.

## 3. FORMULATION OF THE PROBLEM

Next we introduce some geometrical notions, requirements and functional spaces and use them for the formulation of the initial boundary value problem in the case of irregular domains. Several useful results from ref. 14 are also included. We refer to ref. 11 for the detailed observation of the geometric measure theory and to ref. 14 for the discussion of a generalisation of the boundary value conditions and trace properties for the kinetic equations in irregular domains.

We assume that $\Omega$ is a bounded domain in $R^n$ with boundary $\partial\Omega$ having finite $n-1$-dimensional Hausdorff measure: $H^{n-1}(\partial\Omega) < \infty$. $\mathfrak{L}^k$ denotes $k$-dimensional Lebesgue measure.

The Structure Theorem of Federer[11] implies that in the sense of Favard measure $G_1^{n-1}$, for almost all points $x$ on the boundary $\partial\Omega$, the tangent plane and the normal $n(x)$ exist in the sense of some approximation. It means that for almost all directions $s$ in $R^n$ the projection of such subsets $B \in \partial\Omega$ that do not possess approximate tangent plane, on the plane $Q(s)$, orthogonal to $s$, has Favard measure zero. Evidently the same properties are valid for the domain $[0, T] \times \Omega$ in $R^{n+1}$.

Such geometrical properties of the domain $\Omega$ are not enough to determine fluxes for the equations in $\Omega$. But given properties determine for the distribution function $f(t, x, v)$ correct fluxes $\int_{\partial\Omega \times R^n} f\, |v \cdot n|\, dG_1^{n-1}\, dv$ averaged over the velocity $v$.

Let us introduce some notations used below.

- $V = \Omega \times R^n$, $\|f\|_V = \int_V |f(x, v)|\, dx\, dv$.
- $D = [0, T] \times \Omega \times R^n$, $T > 0$. $D_t = [0, t] \times \Omega \times R^n$, $0 < t < T$. $\mathfrak{r} = (t, x, v)$. A measure associated below with $D$ and $D_t$ is the Lebesgue measure $\mathfrak{L}^{2n+1}$.
- $\|f\|_D = \int_D |f(\mathfrak{r})|\, d\mathfrak{r}$ and $\|f\|_{L_r(V)} = \int_V |v|_M^r\, |f(x, v)|\, dx\, dv$, $|v|_M = \max(1, |v|)$,
- $\Sigma^{\pm} = [0, T] \times \{(x, v)\colon x \in \partial\Omega, v \in R^n, \pm n(x) \cdot v \geqslant 0\}$, $\Sigma = \Sigma^+ \cup \Sigma^-$.

• $d\sigma = d\sigma(\mathfrak{r}) = dt\, dG_1^{n-1}\, |v \cdot n|\, dv$ is a measure, associated with $\Sigma^{\pm}$. Below by statements valid almost everywhere on $\Sigma^{\pm}$, we mean $\sigma$-almost everywhere validity.

• $\|f^{\pm}\|_{\Sigma^{\pm}} = \int_{\Sigma^{\pm}} |f^{\pm}(t, x, v)|\, d\sigma$ and
$\|f^{\pm}\|_{L_r(\Sigma^{\pm})} = \int_{\Sigma^{\pm}} |v|_M^r\, |f^{\pm}(t, x, v)|\, d\sigma$.

• For a vector $v \in R^n$, let $Q_t(v)$ be the $R^n$-dimensional plane in $R^{n+1}$ intersecting the point $0 \in R^{n+1}$, and orthogonal to the vector $(1, v)$.

• For a point $(t, y) \in R^{n+1}$ let $l_t(y, v)$ be a straight line in $R^{n+1}$ which intersects the point $(t, y)$ and parallel to the vector $(1, v) \in R^{n+1}$.

• $Pr_v$ is an orthogonal projection from $R^{n+1}$ to $Q_t(v)$ in the direction parallel to $(1, v)$.

• $\text{path}_t(y, v) = ([0, T] \times \Omega) \cap l_t(y, v)$.

• $t^-(\mathfrak{r}_b) = \inf(\{s > 0\colon (t_b + s, x_b + sv, v) \in \Sigma^-\})$, $t^+(\mathfrak{r}_b) = \inf(\{s > 0\colon (t_b - s, x_b - sv, v) \in \Sigma^+\})$, $\mathfrak{r}_b = (t_b, x_b, v)$.

The following lemmas are proved in ref. 14 and give useful tools for the analysis of kinetic equations in domains with irregular boundary.

**Lemma 3.1.** For almost all $v \in R^n$ and one of the following two conditions:

(i) for $\mathfrak{L}^n$-almost all $(t, y) \in Q_t(v)$,

(ii) for $\mathfrak{L}^1 \times G_1^{n-1}$-almost all $(t', x) \in [0, T] \times \partial\Omega$ with $(t, y) = Pr_v(t', x)$, the following statement is valid:

$\text{path}_t(y, v)$ consists of a finite set of segments on the line $l_t(y, v)$ in $R^{n+1}$:

$$\text{path}_t(y, v) = \bigcup_i [a_t^i(y, v), b_t^i(y, v)]. \tag{3.1}$$

**Lemma 3.2.** If $f \in L^1([0, T] \times \Omega \times R^n)$, then for almost all $v \in R^n$ and $\mathfrak{L}^n$-almost all $(t, y) \in Q_t(v)$, it follows that $f|_{\text{path}_t(y, v)}$ is summable and

$$\int_D f(\mathfrak{r})\, d\mathfrak{r} = \int_{R^n} \int_{Q_t(v)} \left[ \sum_i \int_{[a_t^i(y, v),\, b_t^i(y, v)]} f(\mathfrak{r})\, d\mathfrak{L}^1 \right] d\mathfrak{L}^n\, dv \tag{3.2}$$

where $[a, b]$ with $a, b \in R^k$ means a segment in $R^k$ with end points $a$ and $b$.

**Lemma 3.3.**

$$\text{If } \sum_i f(a_t^i(y, v), v)\sqrt{1+v^2} \in L^1(\{(Q_t(v), v)\colon v \in R^n\}), \text{ then}$$

$$\int_{R^n}\int_{Q_t(v)} \sum_i f(a_t^i(y, v), v)\sqrt{1+v^2}\, d\Omega^n\, dv$$
$$= \int_{\Sigma^+} f^+(\mathrm{r})\, d\sigma(\mathrm{r}) + \int_{R^n}\int_{\Omega} f(0, x, v)\, dx\, dv \tag{3.3}$$

The analogous relation is valid for integrals of $f(b_t^i(y, v), v)$, $f^-(t, x, v)$ and $f(T, x, v)$.

The following lemma is proved in ref. 14 and is analogous to a result in ref. 17. The proof in the case of irregular boundaries is analogous to the one in ref. 17, but is based on Lemmas 3.1–3.3 instead of the classical Gauss–Green formula.

**Lemma 3.4.** Suppose that $F \in L^1(D)$ and $(\partial/\partial t)\, F + v\, \nabla_x F \in L^1(D)$.

1. Then $F$ has unique traces $F^\pm$ $\sigma$-almost everywhere on $\Sigma^\pm$.

2. If $F^+$ belongs to $L^1(\Sigma^+)$ and $F(0) \in L^1(V)$, then $F^-$ belongs to $L^1(\Sigma^-)$ and

$$\|F(T)\|_V + \|F^-\|_{\Sigma^-} = \|F(0)\|_V + \|F^+\|_{\Sigma^+} + \int_D \left(\frac{\partial}{\partial t}|F| + v\,\nabla_x |F|\right) d\mathrm{r} \tag{3.4}$$

In the theory of kinetic equations, fluxes used in estimates for non stationary problems are integrals over $Q_t(v) \times R^n$, i.e. they are fluxes projected in velocity directions and averaged both over the time interval, over the projection of the boundary and over the velocity.

As a consequence of Lemmas 3.1–3.3 the subsets of the boundary having zero Favard measure are negligible for kinetic boundary value problems. On the rest of the boundary $G_1^{n-1} = H^{n-1}$, an approximate tangent plane[11] is defined and therefore kinetic boundary conditions are defined $H^{n-1}$-almost everywhere.

The integration by parts along the segments of $\text{path}_t(y, v)$ valid by Lemma 3.3 gives a useful substitution of the classical Gauss–Green formula which is not valid generally in the present setup.

Let us introduce a usual parametrisation of the distribution function

$$f^{\#}(t, \mathrm{r}_b) = f(t_b + t, x_b + tv, v), \qquad \mathrm{r}_b = (t_b, x_b, v) \in D \tag{3.5}$$

for $-t^+(\mathrm{r}_b) \leqslant t \leqslant t^-(\mathrm{r}_b)$ and zero otherwise. $f^\#(t, \mathrm{r}_b)$ is a shift of the distribution function after the time $t_b$ along the collisionless paths of molecules in $(t, x) - R^{n+1}$ space.

$$\frac{\partial}{\partial \tau} f^\#(\tau, \mathrm{r}) = \frac{\partial}{\partial t} f(t+\tau, x+\tau v, v) + v\,\nabla_x f(t+\tau, x+\tau v, v) \qquad (3.6)$$

For functions $f$ constant along the collisionless trajectories and determined by the initial data $f_0$ and the ingoing values $f^+$ we use the notation:

$$f_0^\#(t, \mathrm{r}_b) = f_0(\mathrm{r}_b), \quad \mathrm{r}_b \in \{0\} \times V, \quad \text{and} \quad (f^+)^\#(t, \mathrm{r}_b) = f^+(\mathrm{r}_b), \quad \mathrm{r}_b \in \Sigma^+$$

We use for the Enskog equation a functional space based on the parametrisation of $D$ by the segments of $\text{path}_t(y, v)$. Let us introduce a norm calculating the supremum over each segment $[a_t^i(y, v), b_t^i(y, v)]$ for a.a $(t, y) \in Q_t(v)$ and averaging these suprema by the natural surface measure on the boundary of $D$ over their "ingoing" points $(a_t^i(y, v), v)$. Part of the ingoing points lie on $\{0\} \times V$—the bottom surface of $D$, and the remainder on its lateral surface $\Sigma^+$. The idea of norms calculating the average of suprema of the distribution function over collisionless paths was introduced into nonlinear kinetic theory by Toscani in a discrete velocity setting.[16]

Let for the function $g(\mathrm{r})$, $\mathrm{r} \in D$

$$G_T(\mathrm{r}_b) = \operatorname*{ess\,sup}_{0 \leqslant s + t_b \leqslant T} |g^\#(s, \mathrm{r}_b)|, \qquad \mathrm{r}_b \in D$$

Let $L_{r,T}$ be the space of measurable functions on $[0, T] \times V$ with norm

$$\|g\|_{r,T} = \|G_T(\mathrm{r}_0)\|_{L_r(V)} + \|G_T(\mathrm{r}_+)\|_{L_r(\Sigma^+)}$$

where $\mathrm{r}_0 \in \{0\} \times V$ in the first term and $\mathrm{r}_+ \in \Sigma^+$ in the second.

It is easy to see that the function $G_T(\mathrm{r})$ is constant on collisionless trajectories in $D$ and

$$\frac{\partial G_T(\mathrm{r})}{\partial t} + v\,\nabla_x G_T(\mathrm{r}) = 0$$

Lemma 3.4 implies that

$$\|G_T(\mathrm{r}_0)\|_{L_r(V)} + \|G_T(\mathrm{r}_+)\|_{L_r(\Sigma^+)} = \|G_T(\mathrm{r}_T)\|_{L_r(V)} + \|G_T(\mathrm{r}_-)\|_{L_r(\Sigma^-)} \qquad (3.7)$$

where $\mathrm{r}_T \in \{T\} \times V$ and $\mathrm{r}_- \in \Sigma^-$.

Let us denote by $B^{\pm}f$ the trace of the function $f \in L_{r,T}$ on $\Sigma^{\pm}$, then

$$\|B^{\pm}g\|_{L_r(\Sigma)} \leqslant \|g\|_{r,T} \tag{3.8}$$

It follows from the fact that the supremum over the segments $[a_t^i(y, v), b_t^i(y, v)]$ of the trajectories estimates boundary values.

Denote by $\mathscr{M}(\delta)$ the set of all measurable subsets $M \subset V$ such that for almost all $v \in R^n$ the set $M_v$ of those $x$ for which $(x, v) \in M$ has $\mathfrak{L}^n$ measure less than $\delta$. On such sets the integral form of the Enskog equation has some additional averaging properties. Let

$$\mathscr{I}_T(\delta, f) = \sup_{\mathscr{M}(\delta)} \int_M |f(x, v)| \, dx \, dv$$

Denote by $\mathscr{M}_b(\delta)$ the set of all measurable subsets $M_b \subset \Sigma^+$, such that for almost all $v \in R^n$ the set $M_v$ of those $(t_b, x_b)$ for which $(t_b, x_b, v) \in M_b$, has $\mathfrak{L}^1 \times G_{n-1}^1$ measure less then $\delta$. Let

$$\mathscr{I}_b(\delta, f^+) = \sup_{\mathscr{M}_b(\delta)} \int_{M_b} f^+(t_b, x_b, v) \, d\sigma(\mathfrak{x}_b)$$

For functions defined both on $\{0\} \times V$ and $\Sigma^+$ we introduce

$$\mathscr{I}(\delta, f) = \mathscr{I}_T(\delta, f(0, \cdot)) + \mathscr{I}_b(\delta, f^+) \tag{3.9}$$

A splitting of distribution functions into parts with high and bounded velocities is used below for the construction of the approximate solutions.

Let

$$f_{i0}(x, v) = \min(F_0(x, v), w) \qquad \text{for} \quad |v|^2 \leqslant w^2$$

and $f_{i0} = 0$ otherwise.

Let us introduce the following splitting of the distribution function: $f = f_i + f_e$, where $f_i$ is a solution of the collisionless problem with initial data $f_{i0}$.

$$\frac{\partial}{\partial t} f_i + v \nabla_x f_i = 0, \qquad \text{on } D \tag{3.10}$$

$$f_i^+ = Rf_i^-, \qquad \text{on } \Sigma^+ \tag{3.11}$$

$$f_i|_{t=0} = f_{i0}, \qquad \text{on } V \tag{3.12}$$

The relevant integral form of the equation for $f_e$ is:

$$f_e^{\#}(\tau, \mathrm{r}) = f_e(\mathrm{r}) + \int_0^{\tau} Q^{\#}(f_i + f_e, f_i + f_e)(s, \mathrm{r})\, ds, \qquad \text{on } D \qquad (3.13)$$

$$f_e^+ = R f_e^-, \qquad \text{on } \Sigma^+ \qquad (3.14)$$

$$f_e|_{t=0} = f_{e0}, \qquad \text{on } V \qquad (3.15)$$

The argumentation in the present paper generally follows the method from ref. 3. For the fixed large time interval $[0, T_1]$ we choose $w < 2^{j-1}$ large enough to guarantee such estimates of the initial data $f_{e0}$ and ingoing data $f_e^+$ that they are preserved for $t \in [0, T_1]$. Therefore we concentrate mainly on the developments necessary for the case of the initial boundary value problem and on the methods essential in the case of irregular domains.

## 4. BOUNDARY CONDITIONS AND *A PRIORI* ESTIMATES

The correlation between $f_i$ and $f_e$ in Eq. (3.13) needs relevant estimates for the part of $f_i^+$ with large velocities and for the unbounded part of $f_i^+$.

In the case of the unbounded space and in the periodic case $f_i$ is trivial. It is simply a shift of the initial data $(f_{i0})^{\#}$ bounded, and having compact support in the $v$ variable.

For the case of the Maxwell reflection operator and convex domains, properties of collisionless solutions which we are interested in were proved in ref. 6. Here we present conditions which guarantee the same behaviour of these solutions in the case of irregular domains and general requirements for the operator $R$.

**Condition 4.1.** Boundary conditions are supposed to be of the following type,

$$f^+(t, x, v) = \frac{1}{|n \cdot v|} \int_{n(x) \cdot v' < 0} R(x, v' \to v)\, f^-(t, x, v')\, |n(x) \cdot v'|\, dv' \qquad (4.1)$$

and are defined at $G_1^{n-1}$-almost all points $x$ on $\partial\Omega$. We assume that $R(x, v' \to v)$ is non-negative and

$$R(x, v' \to v) \leqslant \varphi(|v|) \qquad (4.2)$$

$$\int_{n \cdot v > 0} (1 + |v|^m)\, \varphi(|v|)\, dv \leqslant C_m, \qquad m \geqslant 2 \qquad (4.3)$$

The flux of energy and the arbitrary amount of the higher moments for molecules reflected from the boundary are uniformly estimated by the flux of mass of outgoing molecules. The following requirement is essential in the case of irregular geometry of the boundary.

**Condition 4.2.** There is such $C>0$ independent of points on $\partial\Omega$

$$CM(x) \leqslant R(x, v' \to v) \tag{4.4}$$

where $M(x)$ is a Maxwell distribution associated with the point $x \in \partial\Omega$. The temperature $\theta(x)$ is uniformly bounded on $\partial\Omega$ from below and from above.

The following property of the operator $R$ follows from (4.4). Let $\alpha > 0$, $\mathscr{Y}$ is an arbitrary cone in $n \cdot v > 0$ with vertex at zero and body angle larger than $\alpha$. Then for some $\lambda_0 > 0$ uniformly relatively to $x$, $v'$ and $\mathscr{Y}$

$$\int_{\mathscr{Y}} R(x, v' \to v)\, dv \geqslant \lambda_0 \tag{4.5}$$

This property means that a molecule which falls on the surface has a strictly positive probability to be reflected to any body angle larger than $\alpha$ in the half-space.

One type of entropy estimates in a bounded domain can be proved if some additional regularity of the reflecting operator $R$ at the boundary is assumed. In the case of regular boundary this requirement was introduced in ref. 5.

**Condition 4.3.** There is a constant $C_4 < \infty$ and $\alpha_0 \in [0, 1)$ such that

$$\int_{n \cdot v \geqslant 0} Rf^- \ln\left(\frac{Rf^-}{q_0^+}\right) |v \cdot n|\, dv - \alpha_0 \int_{n \cdot v < 0} f^- \ln\left(\frac{f^-}{q_0^-}\right) |v \cdot n|\, dv \leqslant C_4(q_0^- + q_2^-) \tag{4.6}$$

where $q_k^\pm = \int_{R^n} f^\pm\, |v|^k\, |v \cdot n|\, dv$, and the function $f \geqslant 0$ is such that all relevant integrals exist.

Conditions 4.1–4.3 are valid for regular models of reflection and also for Maxwellian diffuse reflection.[5]

Next we introduce a geometric assumption which is used for the specific estimates of the solutions in irregular domains.[14]

**Condition 4.4** (Cone condition). There exist $\delta_0 > 0$ and $\alpha > 0$ such that for $G_1^{n-1}$-almost all points $x \in \partial\Omega$ there exists a cone *Cone* inside $\Omega$ with vertex at $x$, body angle larger than $\alpha$, and length larger than $\delta_0$.

Notice that only the estimates of its body angle and length are important. By a cone we mean an arbitrary set consisting of segments with a common end point.

**Lemma 4.1.** Let Conditions 4.1–4.4 be satisfied. Let $f_{i0}(1 + v^2 + \ln f_{i0}) \in L_0(V)$. Then the solution $f_i \in L_{0,T}$ of the problem (3.10)–(3.12) exists for arbitrary $T > 0$, is unique, conserves mass and has the following properties. Integrals

$$\int_V |v|^2 f_i(t, x, v)\, dx\, dv, \qquad \int_V f_i(t, x, v)\, |\ln f_i(t, x, v)|\, dx\, dv \tag{4.7}$$

are uniformly bounded with respect to $t$ in $R^+$. This also holds for the mass, energy and entropy flows averaged over the arbitrary finite interval of time:

$$\int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} (1 + |v|^2)\, f_i^{\pm}\, d\sigma, \qquad \int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} f_i^{\pm} \ln f_i^{\pm}\, d\sigma \tag{4.8}$$

The estimates (4.8) are dependent of $\Delta t$, but independent of $t$.

*Proof.* The equation for $f_i^+$ on $\Sigma^+$ is

$$f_i^+ = RB^-(f_i^+)^{\#} + RB^-(f_{i0})^{\#} \tag{4.9}$$

Let $A = \{(t, x, v) = (t_b + \tau, x_b + v\tau, v)\colon\ (t_b, x_b) \in [t, t + \Delta t] \times \partial\Omega,\ v \in$ *Cone*, $|v| < \gamma\}$, where *Cone* is the cone from Condition 4.4 and $t_b + \tau \leqslant t + \Delta t$.

Let us denote the set of "ingoing" points of $A$ by $A^+ \in \Sigma^+$ and the set of "outgoing" points of $A$ by $A_V \in \{t + \Delta t\} \times V$. Then

$$\int_{A^+} f_i^+\, d\sigma = \int_{A_V} (f_i^+)^{\#}\, dx\, dv \tag{4.10}$$

Conditions 4.1 and 4.2 imply that for enough large $\gamma$ the following estimate is valid:

$$\int_{R^n} f_i^+\, v \cdot n\, dv \leqslant C \int_{Cone,\ |v| < \gamma} f_i^+\, v \cdot n\, dv \tag{4.11}$$

The estimate (4.11) and the construction of the set $A$ imply that for the time interval $\Delta t < \delta_0/\gamma$, equation for $f_i^+$ can be solved in $L_0(\Sigma^+)$ by a contraction argument, and therefore the solution of the problem (3.10)–(3.12) exists and is unique. For the arbitrary interval of time the existence follows by repeating the contraction argument, from the mass conservation law on the boundary and Lemma 3.4.

Next we will prove the estimate for the entropy flux on the boundary. Lemma 4.1 and Lemma 5.3 from ref. 14 imply that

$$\int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} f_i^{\pm} \left| \ln \frac{f_i^{\pm}}{q} \right| d\sigma < C \qquad \text{with} \quad q = \int_{R^n} f_i^{\pm} \, |v \cdot n| \, dv \tag{4.12}$$

$$\int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} f_i^{\pm}(1 + |v|^2) \, d\sigma < C \tag{4.13}$$

If $f_i^+ < q$ then

$$\int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} f_i^+ \ln_+ f_i^+ \, d\sigma < \int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} f_i^+ \ln_+ q \, d\sigma$$

If $f_i^+ > q$ then the estimate (4.12) implies

$$\int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} (f_i^+ \ln f_i^+ + f_i^+ \ln_- q) \, d\sigma$$

$$< \int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} f_i^+ \ln_+ q \, d\sigma + C$$

The standard estimate of $f \ln_- f$, the estimate (4.13) for fluxes of energy and mass, and the boundedness of $q \ln_- q$ imply

$$\int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} f_i^+ \ln_+ f_i^+ \, d\sigma < \int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} f_i^+ \ln_+ q \, d\sigma + C \tag{4.14}$$

The estimate (4.11) implies:

$$\int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} f_i^+ \ln_+ q \, d\sigma \leqslant C \int_{A^+} f_i^+ \ln_+ q \, d\sigma + C \tag{4.15}$$

Condition 4.4 implies that

$$\int_{A^+} f_i^+ \ln_+(q) \, d\sigma \leqslant \int_{A^+} f_i^+ \ln_+ \left( \frac{f_i^+}{M} \right) d\sigma \tag{4.16}$$

Using the conservation of $f_i$ along the collisionless trajectories and setting $\Delta t = \delta_0/\gamma$, we get:

$$\int_{A^+} f_i^+ \ln_+ \left(\frac{f_i^+}{M}\right) d\sigma = \int_{A_V} f_i \ln_+ \left(\frac{f_i}{M}\right) dx\, dv$$

$$\leqslant \int_V f_i(t+\Delta t) \ln_+ \left(\frac{f_i(t+\Delta t)}{M}\right) dx\, dv \tag{4.17}$$

Estimates (4.14)–(4.17) imply the statement of the lemma for $\Delta t = \delta_0/\gamma$. The result for arbitrary finite $\Delta t$ follows by the repetition of the proof for the sequence of time intervals.

A similar proof gives the existence result and the useful estimates for the solutions of the following boundary value problem:

$$\frac{\partial}{\partial t} f + v\, \nabla_x f + f \cdot \nu = g, \qquad \text{on } D \tag{4.18}$$

$$f^+ = Rf^-, \qquad \text{on } \Sigma^+ \tag{4.19}$$

$$f|_{t=0} = F_0, \qquad \text{on } V \tag{4.20}$$

with the function $\nu \geqslant 0$, $\nu \in L^1_{\text{loc}}(D)$.

**Lemma 4.2.** Let $f_0 \in L_r(V)$, $g \in L_{r,T}$ and Conditions 4.1–4.5 be satisfied. Then the problem (4.18)–(4.20) has a unique solution in $L_{r,T}$ and the following estimates are valid:

$$\int_{[t,\, t+\Delta t]} \int_{\partial\Omega} \int_{R^n} (1 + |v|^r)\, f_i^{\pm}\, d\sigma \leqslant C(\|F_0\|_{L_r(V)} + \|g\|_{L_{r,T}}) \tag{4.21}$$

$$\|f(t)\|_{L_r(V)} \leqslant C(\alpha, \delta_0, \gamma)(\|F_0\|_{L_r(V)} + \|g\|_{L_{r,T}}) \tag{4.22}$$

where constants $C$, $C(\alpha, \delta_0, \gamma)$ depend on the geometrical conditions, on $r$, and on the properties of the operator $R$. If the temperature on the boundary is constant, they are independent on $t$.

## 5. EXISTENCE AND UNIQUENESS IN THE CASE OF BOUNDED VELOCITIES

This section contains the analysis of the Enskog equation with bounded velocities. It uses a modified collision operator and the appropriate integral operator in the integral form of the equation. Let

$$Q^j(f, f) = \sigma^2 \int_{\mathscr{L}_+ \times R^n} (f'f'_- - ff_+) W_j((v - v_*) \cdot u) \, dv_* \, du$$

$$L_j f^\#(t, \mathrm{r}_b) = \sigma^2 \int_0^t ds \int_{\mathscr{L}_+ \times R^n} f(t_b + s, x_b + sv + \sigma u, v_*) \times W_j((v - v_*) \cdot u) \, dv_* \, du$$

where $W_j = 1$ if $v^2 + v_*^2 \leqslant 2^{2j}$, and $W_j = 0$ otherwise. The appropriate integral form of the problem is:

$$f^\#(\tau, \mathrm{r}) = f(\mathrm{r}) + \int_0^\tau Q^j(f, f)^\# (s, \mathrm{r}) \, ds, \qquad \text{on } D \tag{5.1}$$

$$f^+ = Rf^-, \qquad \text{on } \Sigma^+ \tag{5.2}$$

$$f|_{t=0} = F_0, \qquad \text{on } V \tag{5.3}$$

For shorter notations, in the following we omit the domain of integration $\mathscr{L}_+ \times R^n$ in the collision integrals over $dv_* \, du$.

Integrals of the following type are important for the estimates of operators in the integral form of the Enskog equation. We notice that when the space argument is outside of the domain $\Omega$, functions are assumed be equal to zero.

Let the function $F$ be such that $(\partial/\partial t) F + v \nabla_x F = 0$,

$$I_T = \int_0^T ds \int_{R^n} \int_{\mathscr{L}_+} F^\#(s, 0, x_b + s(v - v_*) + \sigma u, v_*) ((v - v_*) \cdot u) \, \sigma^2 W_j \, dv_* \, du \tag{5.4}$$

$$I_b = \int_0^{T - t_b} ds \int_{R^n} \int_{\mathscr{L}_+} F^\#(s, t_b, x_b + s(v - v_*) + \sigma u, v_*) ((v - v_*) \cdot u) \, \sigma^2 W_j \, dv_* \, du, \qquad t_b < T \tag{5.5}$$

**Lemma 5.1:**

$$I_T + I_b \leqslant \mathscr{I}_T(\delta_j, F) + \mathscr{I}_b(\delta_j, F) \tag{5.6}$$

for $\delta_j = T 2^{j+1} \sigma^2 \pi$, $T < \Delta t$ with $\Delta t = \delta_0 / \gamma$, chosen such as in the proof of Lemma 4.1.

*Proof.* The estimate follows from the following observation. The change of variables $\sigma^2((v - v_*) \cdot u)_+ \, ds \, du \to dx$, where $x = x_b + s(v - v_*) + \sigma u$, has Jacobian equal to 1. $T |v - v_*|$ is a length of a cylinder including the domain of integration by $x$, $\sigma^2\pi$ is the area of its bottom. The presence of $W_j$ under the integral implies that $|v-v_*|\leqslant 2w=2^{j+1}$ in the domain of integration. Then we notice that the integral over the set $M$ in $\{t\}\times\Omega$ by $dx$ can be transformed to the integral over the projection of the set $M$ in the direction of the vector $(-1,-v_*)$ in $R^{n+1}$ to $([0,T]\times\partial\Omega)\cup(\{0\})\times\Omega)$ with the appropriate measure at each component.

**Lemma 5.2.** Let

$$Q^-(f_i,f_i)^{\#}(s,\mathfrak{r}_b)=\int f_i^{\#}(s,\mathfrak{r}_b)\,f_i^{\#}(s,t_b,x_b+s(v-v_*)+\sigma u,v_*)\,S\,dv_*\,du \tag{5.7}$$

where $s\in R$, $s>0$, $\mathfrak{r}_b=(t_b,x_b,v)$, and $\mathfrak{r}_b\in\{0\}\times V$ or $\mathfrak{r}_b\in\Sigma^+$, $S=\max(0,((v-v_*)\cdot u))$.

Then for $T<\Delta t$ with $\Delta t=\delta_0/\gamma$

$$\begin{aligned}&\int_V |v|_M^r\,dx_b\,dv\int_0^T ds\,Q^-(f_i,f_i)^{\#}(s,0,x_b,v))\\&\quad+\int_{\Sigma^+}|v|_M^r\,d\sigma(\mathfrak{r}_b)\int_0^{T-t_b}ds\,Q^-(f_i,f_i)^{\#}(s,t_b,x_b,v)\\&\quad\leqslant(\mathscr{I}_T(2w\sigma^2\pi T,f_{i0})+\mathscr{I}_b(2w\sigma^2\pi T,f_i^+))(\|f_{i0}\|\,L_r+\|f_i^+\|_{L_r(\Sigma^+)})\end{aligned} \tag{5.8}$$

*Proof.* Remark that $f_i^{\#}(s,t_b,x_b+s(v-v_*)+\sigma u,v_*)=f_i(t_b+s,x+sv+\sigma u,v_*)$. We here discuss the second term, the arguments for the first term being analogous. First the function with $v$ velocity argument in the collision term is estimated by its supremum over the collisionless trajectories and the integral over $v$ gives the $\|\ \|_{L_r(\Sigma^+)}$ norm. Then the integral over $du\,ds$ is estimated by the integral over $[0,T]\times\partial\Omega\cup\{0\}\times\Omega$ as in Lemma 5.1:

$$\begin{aligned}&\int_{\Sigma^+}|v|_M^r\,d\sigma(\mathfrak{r}_b)\int_0^{T-t_b}ds\int\operatorname*{ess\,sup}_{0\leqslant t_b+s\leqslant T}(f_i(t_b+s,x_b+sv,v))\\&\qquad\times f_i(t_b+s,x_b+s(v-v_*)+sv_*+\sigma u,v_*)\,S\,dv_*\,du\\&\quad\leqslant\|f_i^+\|_{L_r(\Sigma)}\int_0^{T-t_b}ds\int f_i(t_b+s,x_b+s(v-v_*)+sv_*+\sigma u,v_*)\,S\,dv_*\,du\\&\quad\leqslant\|f_i^+\|_{L_r(\Sigma)}(\mathscr{I}_T(2w\sigma^2\pi T,f_{i0})+\mathscr{I}_b(2w\sigma^2\pi T,f_i^+))\end{aligned} \tag{5.9}$$

The same estimate as in (5.8) bounds also the trace $B^-\int_0^t Q^-(f_i,f_i)^{\#}\,ds$ of the operator $\int_0^t Q^-(f_i,f_i)^{\#}\,ds$ on $L_r(\Sigma^-)$:

**Proposition 5.3:**

$$\left\| B^- \int_0^t Q^-(f_i, f_i)^\# \, ds \right\|_{L_r(\Sigma)}$$
$$\leqslant (\mathscr{I}_T(2w\sigma^2\pi T, f_{i0}) + \mathscr{I}_b(2w\sigma^2\pi T, f_i^+))(\|f_{i0}\|_r + \|f_i^+\|_{L_r(\Sigma)}) \tag{5.10}$$

**Lemma 5.4.** If $g \in L_{r,T}^+$ and $T > 0$, then

$$\int_{\Sigma^+} |v|_M^r \, d\sigma(\mathrm{x}) \int_0^{T-t} ds \int g^\#(s, t, x, v)$$
$$\times g^\#(s, t, x + s(v - v_*) + \sigma u, v_*) \, S \, dv_* \, du$$
$$+ \int_V |v|_M^r \, dx \, dv \int_0^T ds \int g^\#(s, 0, x, v)$$
$$\times g^\#(s, 0, x + s(v - v_*) + \sigma u, v_*) \, S \, dv_* \, du$$
$$\leqslant \|g\|_{r,T} \, \|g\|_{0,T} \tag{5.11}$$

*Proof.* The proof follows from the change of variables $\sigma^2((v - v_*) \cdot u))_+ \, du \, ds \to dy$ with $y = x + s(v - v_*) + \sigma u$ and the estimates:

$$\int_0^{T-t} ds \int_{\mathscr{L}_+ \times R^n} |g^\#(s, t, x + s(v - v_*) + \sigma u, v_*) \, S| \, dv_* \, du$$
$$\leqslant \int_V |g^\#(s, t, y, v_*)| \, dv_* \, dy \leqslant \|g\|_{0,T} \tag{5.12}$$

**Theorem 5.5.** Suppose $F_0(1 + v^2) \in L_0^+$, $F_0 \ln F_0 \in L_0$ and Conditions 4.1–4.4 be satisfied.

Then locally in time the problem (5.1)–(5.3) has a unique solution $f^j \in L_0(V)$ that conserves mass.

*Proof.* With $f_0 = 0$ as the initial approximation, define

$$L_j f_n(t) = L_j f_n(t, \mathrm{r}_b)$$
$$= \int_0^t ds \int f_n(t_b + s, x_b + sv + \sigma u, v_*) \, W_j S \, dv_* \, du \tag{5.13}$$

$$f_{n+1}^\#(t, \mathrm{r}_b) = f_{n+1}(\mathrm{r}_b) \exp(-L_j f_n(t)) + \int_0^t ds \exp(-L_j f_n(t) + L_j f_n(s))$$
$$\times \int (f'_n f'_{n*})^\# \, (s, \mathrm{r}_b) \, W_j S \, dv_* \, du \tag{5.14}$$

This is equivalent to the equations

$$f^{\#}_{n+1}(t, \mathrm{r}_b) = f_{n+1}(\mathrm{r}_b) + \int_0^t ds \int (f'_n f'_{n*} - f_{n+1} f_{n*})^{\#} (s, \mathrm{r}_b)\, W_j S\, dv_*\, du \tag{5.15}$$

with $f_{n+1}(\mathrm{r}_b) = F_0$ for $\mathrm{r}_b \in \{0\} \times V$ and $f_{n+1}(\mathrm{r}_b) = f^+_{n+1}$ for $\mathrm{r}_b \in \Sigma^+$ and

$$f^+_{n+1} = R f^-_{n+1} \tag{5.16}$$

We split the successive approximations $f_n$ of solution into the sum of the solution $f_i$ of the free molecular problem with finite initial data $f_{i0}$ having compact support and the deviation ${}_e f_n = f_n - f_i$. Therefore ${}_e f_n$ are successive approximations to $f_e$. The equations for ${}_e f_n$ have the form:

$$\begin{aligned} {}_e f^{\#}_{n+1}(t, \mathrm{r}_b) &= {}_e f_{n+1}(\mathrm{r}_b) + \int_0^t ds \int (f'_n f'_i + f'_i\, {}_e f'_{n*} + {}_e f'_n f'_i + {}_e f'_n\, {}_e f'_{n*} - f_i f_i \\ &\quad - f_i\, {}_e f_{n*} - {}_e f_{n+1} f_i - {}_e f_{n+1}\ {}_e f_{n*})^{\#} (s, \mathrm{r}_b)\, W_j\, S\, dv_*\, du \\ &= {}_e f_{n+1}(\mathrm{r}_b) + \mathscr{I}_1 + \mathscr{I}_2 + \mathscr{I}_3 + \mathscr{I}_4 + \mathscr{I}_5 + \mathscr{I}_6 + \mathscr{I}_7 + \mathscr{I}_8 \end{aligned} \tag{5.17}$$

$${}_e f^+_{n+1} = RB^-\ {}_e f_{n+1} \tag{5.18}$$

where ${}_e f_{n+1}(\mathrm{r}_b) = f_{e0} = F_0 - f_{i0}$ for $\mathrm{r}_b \in \{0\} \times V$ and ${}_e f_{n+1}(\mathrm{r}_b) = {}_e f^+_{n+1}$ for $\mathrm{r}_b \in \Sigma^+$. The terms $\mathscr{I}_1 \cdots \mathscr{I}_8$ represent the terms in the itegral operators in given order.

Inserting the Eq. (5.17) into the boundary conditions (5.18) we get the equation for $f^+_{n+1}$ with the right hand side dependent on ${}_e f_0$ and on the collision integral:

$${}_e f^+_{n+1} = RB^-({}_e f^+_{n+1})^{\#} + RB^-(f^{\#}_{e0} + \mathscr{I}_1 + \cdots + \mathscr{I}_8) \tag{5.19}$$

where $({}_e f^+_{n+1})^{\#}$ denotes a continuation of ${}_e f^+_{n+1}$ constant along collisionless trajectories. The analysis in Lemma 4.1 shows that for $T < \delta_0/\gamma$ the operator ${}_e f^+_{n+1} \to RB^-({}_e f^+_{n+1})^{\#}$ in this equation is contracting in $L_0(\Sigma^+)$. Let us denote by $\mathscr{A}$ the bounded operator solving the Eq. (5.19) in $L_0(\Sigma^+)$. The right hand side of this equation is a sum of two terms: a bounded operator acting from $L_0(V)$ to $L_0(\Sigma^+)$ and a bounded operator acting from $L_{0,T}$ to $L_0(\Sigma^+)$. Therefore equations (5.17) and (5.18) are equivalent to

$$f^{\#}_{n+1}(t, \mathrm{r}_b) = [\mathscr{A} RB^-(f^{\#}_{e0} + \mathscr{I}_1 + \cdots + \mathscr{I}_8)](\mathrm{r}_b) + \mathscr{I}_1 + \cdots + \mathscr{I}_8 \tag{5.20}$$

for $\mathrm{r}_b \in \Sigma^+$.

Estimates (5.8), (5.10) imply:

$$
\begin{aligned}
&\|{}_e f_{n+1}\|_{0,T} \\
&\quad \leqslant (1+\|\mathscr{A}\|_{\Sigma^+}) \\
&\qquad \times (\|f_{e0}\|_V + 2(\mathscr{I}_T(2w\sigma^2\pi T, f_{i0}) \\
&\qquad + \mathscr{I}_b(2w\sigma^2\pi T, f_i^+))(\|f_{i0}\|_V + \|f_i^+\|_{\Sigma^+}) \\
&\qquad + (\|{}_e f_n\|_{0,T} + \|{}_e f_{n+1}\|_{0,T} + 3\mathscr{I}_T(2w\sigma^2\pi T, f_{i0}) \\
&\qquad + 3\mathscr{I}_b(2w\sigma^2\pi T, f_i^+))\,\|{}_e f_n\|_{0,T} \\
&\qquad + (\mathscr{I}_T(2w\sigma^2\pi T, f_{i0}) + \mathscr{I}_b(2w\sigma^2\pi T, f_i^+))\,\|{}_e f_{n+1}\|_{0,T})
\end{aligned} \tag{5.21}
$$

The estimates of the mass, energy and entropy fluxes on $\Sigma^+$ for $f_i$ in Section 3 imply that $f_i^+$ can be splitted into two parts: one part bounded by $w$ and having compact support $|v|^2 \leqslant w^2$, and a singular part $f_{is}^+$ such that by the choice of $w$ its norm $\|f_{is}^+\|_{\Sigma^+}$ in $L_0(\Sigma^+)$ can be made arbitrarily small.

We remark that $\mathscr{I}_b(\mu, f_{is}) \leqslant \|f_{is}^+\|_{\Sigma^+}$. Therefore, we can choose $w$ and $T < \Delta t$ such that:

$$
\begin{aligned}
&(1+\|\mathscr{A}\|_{\Sigma^+})(\mathscr{I}(2w\sigma^2\pi T, \mathscr{B}) + \mathscr{I}_b(2w\sigma^2\pi T, f_{is}^+)) < 16^{-1} \\
&(1+\|\mathscr{A}\|_{\Sigma^+})\, 2(\mathscr{I}(2w\sigma^2\pi T, \mathscr{B}) + \mathscr{I}_b(2w\sigma^2\pi T, f_{is}^+)) \\
&\qquad \times (\|\mathscr{B}\|_V + \|f_i^+\|_{\Sigma^+})) < 16^{-1}
\end{aligned}
$$

for $\mathscr{B}(x, v) \leqslant w$, $|v| \leqslant w$. With the constant $C(\Omega, R)$ in (2.24) taken as $C(\Omega, R) = (1+\|\mathscr{A}\|_{\Sigma^+})^{-1}$ the last estimates imply that $\|{}_e f_{n+1}\|_{0,T} \leqslant 16^{-1}$.

Taking a difference between approximations for $f_e$ of orders $n+1$ and $m+1$, we get:

$$
\begin{aligned}
&\|{}_e f_{n+1} - {}_e f_{m+1}\|_{0,T} \\
&\quad \leqslant (1+\|\mathscr{A}\|_{\Sigma^+}) \\
&\qquad \times (3(\mathscr{I}_T(2w\sigma^2\pi T, f_{i0}) + \mathscr{I}_b(2w\sigma^2\pi T, f_i^+))\,\|{}_e f_n - {}_e f_m\|_{0,T} \\
&\qquad + (\mathscr{I}_T(2w\sigma^2\pi T, f_{i0}) + \mathscr{I}_b(2w\sigma^2\pi T, f_i^+))\,\|{}_e f_{n+1} - {}_e f_{m+1}\|_{0,T} \\
&\qquad + (\|{}_e f_n\|_{0,T} + \|{}_e f_m\|_{0,T} + \|{}_e f_{m+1}\|)\,\|{}_e f_n - {}_e f_m\|_{0,T} \\
&\qquad + \|{}_e f_n\|_{0,T}\,\|{}_e f_{n+1} - {}_e f_{m+1}\|_{0,T}) \\
&\quad \leqslant \tfrac{1}{4}\|{}_e f_{n+1} - {}_e f_{m+1}\|_{0,T} + \tfrac{1}{2}\|{}_e f_n - {}_e f_m\|_{0,T}
\end{aligned} \tag{5.22}
$$

Therefore, $\{{}_e f_n\}_f^\infty$ is a Cauchy sequence in the $\| \|_{0,T}$ norm. The limit $f_e^j$ gives a unique nonnegative solution $f^j = f_i + f_e^j$ for the Enskog equation with bounded velocities. The solution is stable with respect to the perturbation of the initial data in $L_0(V)$.

Next we formulate a regularity result useful for a strict proof of the $H$-theorem for the Enskog equation.

**Proposition 5.6.** Suppose the function $F_0 \in L_0(V)$ is such that $v\nabla_x F_0 \in L_0(V)$ and $Q^j(F_0, F_0) \in L_0(V)$. Then the solution $f$ given by Theorem 5.5 exists on a small time interval $[0, T]$, and $(\partial/\partial t) f \in L_{0,T}$ and $v\nabla_x f \in L_{0,T}$.

The proposition follows immediately from Theorem 5.5 because the assumed conditions imply that $f_t = \partial f/\partial t$ satisfies the following equations:

$$\frac{\partial}{\partial t} f_t + v\nabla_x f_t = Q^j(f_t, f_t), \qquad \text{on } D \tag{5.23}$$

$$f_t^+ = Rf_t^-, \qquad \text{on } \Sigma^+ \tag{5.24}$$

$$f_t|_{t=1} = Q^j(F_0, F_0) - v\nabla_x F_0, \qquad \text{on } V \tag{5.25}$$

Now to get a strict proof of the $H$-theorem and Lemma 2.1 at first for the Enskog equation with bounded velocities we approximate the arbitrary initial data in $L_0(V)$ by the smooth function, satisfying assumptions of the proposition and estimated from below by some Maxwellian. For such initial data the proofs of the $H$-theorem and Lemma 2.1 for the Enskog equation with bounded velocities are valid in a strict sense. The result for the general initial data follows by approximation and the continuous dependence of solutions on initial data.

## 6. ESTIMATES FOR HIGH MOMENTS

The following lemmas contain estimates for various terms of the integral form of the initial boundary value problem for the Enskog equation analogous to estimates in ref. 3 for the Cauchy problem. The proofs in ref. 3 are based on a delicate splitting of the $v \times v_*$ and $v' \times v'_*$ spaces $R^{2n}$ with different types of estimates in the subdomains of this splitting for the velocity variables. The realisation of this idea is independent of the domain $\Omega$ of space variables. The generalisation to the initial boundary value problem with the norms $\| \|_{r,T}$ and the functionals $\mathcal{J}_T(\mu, \cdot)$ and $\mathcal{J}_b(\mu, \cdot)$ introduced in the present paper follows by a combination of arguments from ref. 3 and from the proofs of Lemmas 5.2–5.4. We give in the present section only a proof for Lemma 6.2. The other estimates are proved in a similar way.

We notice that $\mathscr{J}(\mu, F) = \mathscr{J}_T(\mu, F) + \mathscr{J}_b(\mu, F)$.

**Lemma 6.1.** Suppose that $w2^{-k} \leqslant \delta \ll \eta \ll 1$, and that $g \in L^+_{r,T}$. Then the following estimate holds:

$$\int_{\Sigma^+,\, |v| \geqslant 2^{k+1}} d\sigma(\mathrm{r}_b)\, |v|^r_M \int_0^{T-t_b} dt \int (f'_i g'_* + g' f'_{i*})^\# (t, \mathrm{r}_b)\, S\, dv_*\, du$$
$$+ \int_{V,\, |v| \geqslant 2^{k+1}} dx_b\, dv\, |v|^r_M \int_0^T dt \int (f'_i g'_* + g' f'_{i*})^\# (t, 0, x_b, v)\, S\, dv_*\, du$$
$$\leqslant C\{(1-\delta)^{-r} \eta^2 + (1 + \eta^2 - \delta^2)^{-r/2}\}\, \|g\|_{r,T} \tag{6.1}$$

Here $C$ depends on $w$ but not on $f_0$, $k$, or $T$.

**Lemma 6.2.**

$$\int_{\Sigma^+,\, |v| \leqslant 2^{k+1}} d\sigma(\mathrm{r}_b)\, |v|^r_M \int_0^{T-t_b} dt \int (f'_j g'_* + g' f'_{j*})^\# (t, \mathrm{r}_b)\, S\, dv_*\, du$$
$$+ \int_{V,\, |v| \leqslant 2^{k+1}} dx_b\, dv\, |v|^r_M \int_0^T dt \int (f'_j g'_* + g' f'_{j*})^\# (t, 0, x_b, v)\, S\, dv_*\, du$$
$$\leqslant 2^{r/2+1} \mathscr{J}(T 2^{k+3} \pi\sigma^2, F_{iT} |v|^r_M)\, \|g\|_{r,T} + 2^{-r+1}\, \|f_i\|_{0,T}\, \|g\|_{r,T} \tag{6.2}$$

*Proof.* By the symmetry of the estimates under the change of $v' \leftrightarrow v'_*$, we treat only the term with $f'_j g'_*$. We remark that $dv\, dv_* = dv'\, dv'_*$ and split the domain of integration into two parts, $|v'_*| \leqslant 2^{k+2}$ and $|v'_*| \geqslant 2^{k+2}$.

The first part of the integral is estimated by

$$2^{r/2+1} \mathscr{J}(T 2^{k+3} \pi\sigma^2, F_{iT} |v|^r_M)\, \|g\|_{r,T} \tag{6.3}$$

We use the change of variables $\sigma^2((v - v_*) \cdot u)_+\, ds\, du \to dx$, in a small cylindrical neighbourhood in $\Omega$, where $x = x_b + s(v - v_*) + \sigma u$. The volume of such a domain is estimated by $T 2^{k+3} \pi\sigma^2$ because

$$|v - v_*| \leqslant |v|' + |v'_*| \leqslant 2^{k+3}$$

The energy conservation law implies the following sequence of estimates:

$$|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$$

$$|v|_M^2 \leqslant |v'|_M^2 + |v'_*|_M^2$$

$$|v| \leqslant 1 \Rightarrow |v|_M = 1$$

$$|v| \geqslant 1 \Rightarrow 1 \leqslant |v|_M^2$$

$$|v|_M^r \leqslant 2^{r/2} |v'|_M^r |v'_*|_M^r$$

The last inequality together with arguments from the proof of Lemma 5.2 imply the estimate for the first integral.

Substituting $|v|_M$ by $|v_*|_M/2$ into the second integral analogously as in the proof of Lemma 5.4 we get the bound:

$$2^{-r} \|f\|_{0,T} \|g\|_{r,T}$$

**Lemma 6.3.** For $g, p \in L^+_{r,T}$, $0 < \delta \ll 1$, and

$$G_T = \sup_{0 \leqslant t + t_b \leqslant T} g^{\#}(t, \mathrm{r}_b), \qquad P_T = \sup_{0 \leqslant t + t_b \leqslant T} p^{\#}(t, \mathrm{r}_b)$$

for $\mathrm{r}_b \in \Sigma^+$ and $\mathrm{r}_b \in \{0\} \times V$, the following estimate holds:

$$\begin{aligned}
&\int_0^T dt \int_V dx\, dv \int g' p'_* |v|^r S\, dv_*\, du \\
&\quad \leqslant (1 + (1-\delta)^{-r} + \varepsilon\, \delta^{-1/2}\, 2^{r/2})(\|g\|_{r,T} \|p\|_{1/2,T} + \|g\|_{1/2,T} \|p\|_{r,T}) \\
&\qquad + 2^{r/2}(\mathscr{J}(\mu, G_T) \|p\|_{r,T} + \mathscr{J}(\mu, P_T) \|g\|_{r,T})
\end{aligned} \tag{6.4}$$

where $\mu = T 2\pi\varepsilon^{-2}\sigma^2$.

**Lemma 6.4.** Suppose $g, p \in L_{r,T}$ and $0 \leqslant \delta \ll 1$. Then the following estimates hold:

$$\begin{aligned}
&\int_0^T dt \int_V dx\, dv \int g' p'_* (|v|_M^r + |v_*|_M^r) S\, dv_*\, du \\
&\quad \leqslant 2^{r/2+1} \varepsilon^2 \|g\|_{r+1,T} \|p\|_{r,T} + 2 \|g\|_{0,T} \|p\|_{r,T} \\
&\qquad + 2(1-\delta)^{-r} \|g\|_{0,T} \|p\|_{r,T} \\
&\qquad + \varepsilon^{-2r} \delta^{-r} 2^{r/2+1} \mathscr{J}(\delta^{-1}\mu, G_T) \|p\|_{r,T}
\end{aligned} \tag{6.5}$$

$$\int_0^T dt \int_V dx\, dv \int gp_*(|v|_M^r + |v_*|_M^r)\, S\, dv_*\, du$$

$$\leqslant 2\varepsilon^2 \|g\|_{r+1,T} \|p\|_{r,T} + 2\|g\|_{0,T} \|p\|_{r,T}$$

$$+ 2\varepsilon^{-2r} \mathscr{I}(\mu, G_T) \|p\|_{r,T} \tag{6.6}$$

## 7. GLOBAL EXISTENCE AND UNIQUENESS IN THE CASE OF UNBOUNDED VELOCITIES

In this section the solution of the full Enskog equation is obtained as a strong limit of the solution $f^j$ from Theorem 5.5. The main result of the present paper is the following.

**Theorem 7.1.** Suppose that $F_0 \geqslant 0$ is such that $F_0 \ln F_0 \in L_0(V)$, $F_0 \in L_r(V)$ for all $r \geqslant 0$, Conditions 4.1–4.4 are satisfied, and the temperature on the boundary is constant.

Then the Enskog equation has a unique positive solution $f$ with $\|f\|_{r,T} < \infty$ for $r$, $T > 0$ and satisfying (2.21), (2.22), (2.23).

*Proof.* The proof of the theorem begins from the solution of the equation for the ingoing distribution function $f_e^+$.

Let $[0, T_1]$ be a large time interval and choose $w$ so that (2.24) holds. Set

$$(A_j' f)^{\#}(t, \mathrm{r}_b) = \int_0^t \exp(-L_j f_i(t) + L_j f_i(s)) \Big( Q^j(f_i + f, f_i + f)^{\#}(s, \mathrm{r}_b)$$

$$+ \int (f f_{i*})^{\#}(s, \mathrm{r}_b)\, W_j S\, du\, dv \Big)\, ds \tag{7.1}$$

The function $f_e^j = f^j - f_i$ satisfies

$$(f_e^j)^{\#}(t, \mathrm{r}_b) = f_e^j(\mathrm{r}_b) \exp(-L_j f_i(t)) + (A_j' f_e^j)^{\#}(t, \mathrm{r}_b) \tag{7.2}$$

with $f_e^j(\mathrm{r}_b) = f_{e0}(x_b, v)$ for $\mathrm{r}_b \in \{0\} \times V$ and $f_e^j(\mathrm{r}_b) = f_e^{j+}(\mathrm{r}_b)$ for $\mathrm{r}_b \in \Sigma^+$ and

$$f_e^{j+} = R f_e^{j-} \tag{7.3}$$

$f_e^{j+}$ satisfies the equation

$$f_e^{j+} = RB^-(f_e^{j+})^{\#} \exp(-L_j f_i(t)) + RB^-((f_{e0})^{\#} \exp(-L_j f_i(t)) + (A'_j f_e^j)^{\#}) \tag{7.4}$$

Lemma 4.2 implies that Eq. (7.4) for $f_e^{j+}$ has a unique solution in $L_r(\Sigma^+)$. It can be solved first in $L_0(\Sigma^+)$ by a contraction argument on the interval of time $T \leqslant \delta_0/\gamma$. For the right hand side term $g$ belonging to $L_r(\Sigma^+)$, Condition 4.1 implies that $\|f_e^{j+}\|_{\Sigma^+} \leqslant C \|f_e^{j+}\|_{L_0(\Sigma^+)} + \|g\|_{L_r(\Sigma^+)}$.

Let us denote by $\mathscr{A}_L$ a bounded operator solving the Eq. (7.4) in $L_r(\Sigma^+)$. Then for $\mathrm{r}_b \in \Sigma^+$ (7.2) is equivalent to

$$\begin{aligned}(f_e^j)^{\#}(t, \mathrm{r}_b) = \mathscr{A}_L\{&RB^-(f_{e0})^{\#} \exp(-L_j f_i(t)) \\ &+ RB^-(A'_j f_e^j)\}(\mathrm{r}_b) \exp(-L_j f_i(t)) \\ &+ (A'_j f_e^j)^{\#}(t, \mathrm{r}_b), \qquad \mathrm{r}_b \in \Sigma^+\end{aligned} \tag{7.5}$$

By using Condition 4.1 and the fact that norms $\|\mathscr{A}_L\|_{L_r(\Sigma^+)}$ are bounded for all $r$, the scheme of the proof from ref. 3 can be adapted to this initial boundary value problem.

The following lemmas generalise analogous results from ref. 3 to the case of bounded domains.

**Lemma 7.2.** For any $f \in L_{2,T}$ with

$$\sup_{t \leqslant T} \int_V |f(x, v, t)| \, |v|_M^2 \, dx \, dv \leqslant 4 \int_V F_0(x, v) \, |v|_M^2 \, dx \, dv$$

and $0 \leqslant r \leqslant 1$ the following estimate holds:

$$\begin{aligned}\|B^- A'_j f\|_{L_r(\Sigma^-)} \leqslant \|A'_j f\|_{r,T} \leqslant 4(&\mathscr{I}_T(2w\pi\sigma^2 T, f_{i0}) \\ &+ \mathscr{I}_b(2w\pi\sigma^2 T, f_i^+)) \|f_{i0}\|_{r,T} \\ &+ 4\|f\|_{0,T} \|f\|_{r,T} + CT\|F_0\|_{L_2(V)}\end{aligned} \tag{7.6}$$

The following statement gives the uniform estimate for the first moment of the approximate solution $f^j$ and follows from Lemma 7.2 and from the energy and the entropy flux estimates of Lemma 4.1.

**Proposition 7.3.** $T$ in Lemma 7.2 can be chosen such that

$$(1 + \|\mathscr{A}_L\|_{L_1(\Sigma^+)})\, CT\, \|f_0\|_{L_2(V)} < 256^{-1} \tag{7.7}$$

$$(1 + \|\mathscr{A}_L\|_{L_1(\Sigma^+)})(\mathscr{J}(2w\pi\sigma^2 T, \mathscr{B}) + \mathscr{J}_b(2w\pi\sigma^2 T, f_{is}^+))(4\,\|\mathscr{B}\|_{r,T} + 1) < 256^{-1} \tag{7.8}$$

and the solution $f^j$ of Theorem 5.5 exists on $[0, T]$ for every $j$, and

$$(1 + \|\mathscr{A}_L\|_{L_1(\Sigma^+)})\, \|f_e^j\|_{1,T} \leqslant 32^{-1} \tag{7.9}$$

The estimate is valid also for $f_0 = g$ such that (2.21)–(2.24) are satisfied by $f(\cdot, t) = g$, $t < T_1$, with $C(\Omega, R) = (1 + \|\mathscr{A}_L\|_{L_1(\Sigma^+)})^{-1}$.

**Lemma 7.4.** Given $\delta$, there are $\tilde{T} < \Delta t$ and $\mu > 0$ depending on $w$ and $f_0$ but independent on $j$ such that

$$\mathscr{J}_T(\mu, F_{e,T'}^j) + \mathscr{J}_b(\mu, F_{e,T'}^j) \leqslant \delta, \qquad T' \leqslant \tilde{T} \tag{7.10}$$

where $F_{e,T'}^j(\mathfrak{r}_b) = \sup_{0 \leqslant t + t_b \leqslant T'} (f_e^j)^{\#}(t, \mathfrak{r}_b)$.

*Proof.* It is clear that the lemma holds, if $F_{e,T'}^j$ is replaced by $\mathscr{B}$ on $\{0\} \times V$, and by $\mathscr{B} + f_{is}^+$ on $\Sigma^+$. Then $F_{e,T'}^j \leqslant \mathscr{B} + f_{is}^+ + F_{T'}^j$ on $\Sigma^+$, and $F_{e,T'}^j \leqslant \mathscr{B} + F_{T'}^j$ on $\{0\} \times V$. Therefore it is enough to proof the estimate for $\mathscr{J}(\mu, F_{T'}^j)$. Let

$$\begin{aligned}(\mathscr{T}'_1 + \mathscr{T}'_2 + \mathscr{T}'_3 + \mathscr{T}'_4)(\mathfrak{r}_b) \\ = \int_0^{T' - t_b} ds \int (f'_i f'_{i*} + |f'_i f^{j'}_{e*}| + |f^{j'}_e f'_{i*}| + |f^{j'}_e f^{j'}_{e*}|)^{\#}(s, \mathfrak{r}_b)\, W_j S\, dv_*\, du\end{aligned} \tag{7.11}$$

Excluding negative terms from the Eqs. (7.2), (7.5), and using the monotonicity of $\mathscr{A}_L$ we get:

$$\begin{aligned}F_{T'}^j(\mathfrak{r}_b) \leqslant{}& \mathscr{A}_L\{RB^- F_0^{\#} + RB^-(\mathscr{T}'_1 + \cdots + \mathscr{T}'_4)\}(\mathfrak{r}_b) \\ &+ (\mathscr{T}'_1 + \cdots + \mathscr{T}'_4)(\mathfrak{r}_b), \qquad \mathfrak{r}_b \in \Sigma^+\end{aligned} \tag{7.12}$$

$$F_{T'}^j(\mathfrak{r}_b) \leqslant F_0(x_b, v) + (\mathscr{T}'_1 + \mathscr{T}'_2 + \mathscr{T}'_3 + \mathscr{T}'_4)(\mathfrak{r}_b), \qquad \mathfrak{r}_b \in \{0\} \times V \tag{7.13}$$

The terms independent of $j$ satisfy the lemma. Also for terms with $\mathscr{T}_2$ and $\mathscr{T}_3$ the bound follows from arguments as in the proof of Lemma 7.2. Let $\mu = 2^{k+2}\pi\sigma^2\tilde{T}$.

For $\mathrm{r}_b \in \Sigma^+$,

$$\int_{\Sigma^+} \mathscr{T}'_4(\mathrm{r}_b)\, d\sigma(\mathrm{r}_b)$$

$$= \int_{\Sigma^+} d\sigma(\mathrm{r}_b) \int_0^{T'-t_b} ds \int |f_e^{j'} f_{e*}^{j'}|^{\#}\,(s, \mathrm{r}_b)\, W_j S\, dv_*\, du$$

$$\leqslant \int_{\Sigma^+} d\sigma(\mathrm{r}_b) \int_0^{T'-t_b} ds \int |f_e^{j'} f_{e*}^{j'}|^{\#}\,(s, \mathrm{r}_b)$$

$$\times W_{2k} S\, dv_*\, du + 2^{-k+2}\, \|f_e^j\|_{1,T'}\, \|f_e^j\|_{0,T'}$$

$$\leqslant 2^{-k+2}\, \|f_e^j\|_{1,T'}\, \|f_e^j\|_{0,T'} + \sup_{\mathscr{M}_b(\mu)} \int_{M_b} |F^j_{e,T'}|\, d\sigma(\mathrm{r}_b)\, \|f_e^j\|_{0,T'} \tag{7.14}$$

For $\mathrm{r} \in 0 \times V$, the following estimate follows:

$$\int_V \mathscr{T}'_4(\mathrm{r})\, dx\, dv \leqslant 2^{-k+2}\, \|f_e^j\|_{1,T'}\, \|f_e^j\|_{0,T'}$$

$$+ \sup_{\mathscr{M}(\mu)} \int |F^j_{e,T'}|\, dx\, dv\, \|f_e^j\|_{0,T'} \tag{7.15}$$

Integrating inequalities (7.12) and (7.13) over $M \in V$ and $M_b \in \Sigma^+$, taking the supreme over $M \in \mathscr{M}(\mu)$ and $M_b \in \mathscr{M}_b(\mu)$, and moving the terms with $|F^j_{e,T'}|$ to the left-hand side, we get the bound which implies the statement of the lemma:

$$1/2 \left( \sup_{\mathscr{M}_b(\mu)} \int_{M_b \in \Sigma^+} F^j_{eT'}\, d\sigma(\mathrm{r}_b) + \sup_{\mathscr{M}(\mu)} \int_{M \in V} F^j_{eT'}\, dx\, dv \right)$$

$$\leqslant (1 + \|\mathscr{A}_L|_{L_1(\Sigma^+)}) \left\{ \sup_{\mathscr{M}(\mu)} \int_{M \in V} F_0\, dx\, dv + 2^{-k-8} \right.$$

$$+ \sup_{\mathscr{M}_b(\mu)} \int_{M_b \in \Sigma^+} (\mathscr{T}'_1 + \mathscr{T}'_2 + \mathscr{T}'_3)(\mathrm{r}_b)\, d\sigma(\mathrm{r}_b)$$

$$\left. + \sup_{\mathscr{M}(\mu)} \int_{M \in V} (\mathscr{T}'_1 + \mathscr{T}'_2 + \mathscr{T}'_3)(\mathrm{r})\, dx\, dv \right\}$$

The following lemma is a consequence of results from Section 6, Lemmas 7.2, 7.4, and Proposition 7.3.

**Lemma 7.5.** Let $f \in L_{r,T}$. Then the following estimate is valid.

$$\begin{aligned}\|A_j' f\|_{r,T} \leqslant\ & 2^{r/2+2}(\mathscr{I}(2w\pi\sigma^2 T, \mathscr{B}) + \mathscr{I}_b(2w\pi\sigma^2 T, f_{is}^+))\,\|f_{i0}\|_r \\ & + C[(1-\delta)^{-r}\eta^2 + (1+\eta^2-\delta^2)^{-r/2} \\ & + 2^{r/2+2}(\mathscr{I}(2^{k+3}\pi\sigma^2 T, \mathscr{B}\,|v|_M^r) + \mathscr{I}_b(2^{k+3}\pi\sigma^2 T, f_{is}^+\,|v|_M^r)) \\ & + 2^{-r}\tilde{w} + 2^{r/2+1}\mathscr{I}(2\pi\varepsilon^{-2}\sigma^2 T, F_T)]\,\|f\|_{r,T} \\ & + (3 + 2(1-\delta)^{-r} + \varepsilon\delta^{-1/2}2^{r/2+1})\,\|f\|_{1/2,T}\,\|f\|_{r,T}, \quad r \geqslant 1 \qquad (7.16)\end{aligned}$$

with $w2^{-k} \leqslant \delta \ll \eta \ll 1$, $\tilde{w}$, $C$ dependent only on $w$.

For the case of the whole space it is proved in ref. 3 that the estimates having the structure as in Lemmas 7.2–7.5 imply useful bounds for $f^j$ uniform with respect to $j$. In the case of the initial boundary value problem the analogous result follows from the regularity property (4.1) of the reflection operator $R$ and the boundedness of $\|\mathscr{A}_L\|_{L_r(\Sigma^+)}$. The additional averaging given by the operator $R$ in the term

$$\mathscr{A}_L\{RB^-(A_j' f_e^j)\}(\mathrm{r}_b)\exp(-L_j f_i(t)) \qquad (7.17)$$

from the integral form of the Enskog Eq. (7.5) and the argumentation analogous to the proof of Lemma 7.2 imply that the $\|\ \|_{r,T}$ norm of this term is bounded by the value

$$\begin{aligned}TC(r)[&4(\mathscr{I}_T(2w\pi\sigma^2 T, f_{i0}) + \mathscr{I}_b(2w\pi\sigma^2 T, f_i^+))\,\|f_{i0}\|_{1,T} \\ & + 4\,\|{}_e f^j\|_{0,T}\,\|{}_e f^j\|_{1,T} + CT\,\|F_0\|_{L_2(V)}] \qquad (7.18)\end{aligned}$$

This estimate and Lemmas 7.2–7.5 imply the following proposition.

**Proposition 7.6** (Uniform bound). The values of constants $\eta$, $r = r_0$, $\delta$, $k$, $\varepsilon$ from Lemma 7.5 and $\tilde{T}$ from Lemma 7.5 can be chosen such that for $T \leqslant \tilde{T}$ with $\tilde{T}$ and $r_0$ dependent only on $w$ and independent on $j$

$$\|f_e^j\|_{r_0,T} \leqslant C(r_0)(\|f_{e0}\|_{L_{r_0}(V)} + 256^{-1}) \qquad (7.19)$$

For $r > r_0$ there is $\tilde{T}_r$ depending only on $w$ and $r$ such that for $T_r \leqslant \tilde{T}_r$

$$\begin{aligned}\|f_e^j\|_{r-1,T_r} &\leqslant C(r)(\|f_{e0}\|_{L_{r-1}(V)} + 256^{-1}) \\ \|f_e^j\|_{r,T_r} &\leqslant C(r)(\|f_{e0}\|_{L_r(V)} + 256^{-1})\end{aligned} \qquad (7.20)$$

These estimates are valid also for $F_0 = g$ when $g = f(t, \cdot)$, $0 \leqslant t \leqslant T_1$ satisfies (2.21)–(2.24).

We will prove that the sequence $\{f_e^j\}$ has the Cauchy property in $L_{r-1,\tilde{T}}$ for some $\tilde{T} \leqslant \tilde{T}_r$ with $r = r_0$. For $j_1 > j$ consider

$$
\begin{aligned}
&\|f_e^{j_1} - f_e^j\|_{r-1,T'} \\
&\quad \leqslant \|A'_{j_1} f_e^{j_1} - A'_j f_e^j\|_{r-1,T'} \\
&\qquad + \|\mathscr{A}_L\|_{L_{r-1}(\Sigma^+)} \|(RB^-(A'_{j_1} f_e^{j_1} - A'_j f_e^j))^{\#}\|_{r-1,T'} \\
&\quad \leqslant (\|A'_{j_1} f_e^{j_1} - A'_j f_e^{j_1}\|_{r-1,T'} + \|A'_j f_e^{j_1} - A'_j f_e^j\|_{r-1,T'}) \\
&\qquad + \|\mathscr{A}_L\|_{L_{r-1}(\Sigma^+)} \|(RB^-(A'_{j_1} f_e^{j_1} - A'_{j_1} f_e^j))^{\#}\|_{r-1,T'} \\
&\qquad + \|\mathscr{A}_L\|_{L_{r-1}(\Sigma^+)} \|(RB^-(A'_j f_e^{j_1} - A'_j f_e^j))^{\#}\|_{r-1,T'} \qquad (7.21)
\end{aligned}
$$

Lemma 7.5 implies that the term $\|A'_j f_e^{j_1} - A'_j f_e^j\|_{r-1,T'}$ is bounded by

$$
\begin{aligned}
&C[(1-\delta)^{-r}\eta^2 + (1+\eta^2-\delta^2)^{-r/2} + 2^{r/2+2}\mathscr{I}(2^{k+3}\pi\sigma^2 T', \mathscr{B}\,|v|^r_M) \\
&\quad + 2^{r/2+2} + \mathscr{I}_b(2^{k+3}\pi\sigma^2 T', f^+_{is}\,|v|^r_M) + 2^{-r+1}\tilde{w}]\,\|f_e^{j_1} - f_e^j\|_{r,T'} \\
&\quad + \left\| \int_0^t ds \int \{|(f_e^{j_1})'(f_{e*}^{j_1})' - (f_e^j)'(f_{e*}^j)'| + |(f_e^{j_1})(f_{e*}^{j_1}) - (f_e^j)(f_{e*}^j)|\}^{\#} \right. \\
&\quad \left. \times W_j S\,du\,dv_* \right\|_{r-1,T'}
\end{aligned}
$$

Considerations analogous to Lemma 7.5 give that the last term nonlinear with respect to $f_e$ can be bounded by

$$
\begin{aligned}
&\|f_e^{j_1} - f_e^j\|_{r-1,T'} \{2^{r/2+2}\varepsilon^2(\|f_e^j\|_{r-1,T'} + \|f_e^{j_1}\|_{r-1,T'}) \\
&\quad + (4 + 2(1-\delta)^{1-r})(\|f_e^j\|_{0,T'} + \|f_e^{j_1}\|_{0,T'}) \\
&\quad + (\mathscr{I}(\delta^{-1}\mu, F^{j_1}_{e,T'}) + \mathscr{I}(\delta^{-1}\mu, F^j_{e,T'}))\,2^{r/2+2}\varepsilon^{-2r}\delta^{-r}\}
\end{aligned}
$$

where $\mu = T' 2\pi\varepsilon^{-2}\sigma$. The factor $2^{r/2+2}\varepsilon^2(\|f_e^j\|_{r-1,T'} + \|f_e^{j_1}\|_{r-1,T'})$ can be made suitably small by a suitable choice of $\varepsilon$.

Lemma 7.2 implies that for enough small $T' = \tilde{T} > 0$, depending only on $w$, the factor

$$
(\mathscr{I}(\delta^{-1}\mu, F^{j_1}_{e,T'}) + \mathscr{I}(\delta^{-1}\mu, F^j_{e,T'}))
$$

can be maid suitably small uniformly with respect to $f^j$ with initial value $f(T)$, $0 \leqslant T \leqslant T_1$. Therefore $\|A'_j f_e^{j_1} - A'_j f_e^{j_1}\|_{r-1,T'} \leqslant \|f_e^{j_1} - f_e^j\|_{r-1,T'}/2$.

The term $\sup_{j_1 \geqslant j} \|A'_{j_1} f_e^{j_1} - A'_j f_e^{j_1}\|_{r-1,T'} \xrightarrow[j\to\infty]{} 0$. The proof is analogous to one in ref. 3 and uses estimates for the integral form of the

Enskog equation and a splitting of velocity space for large and bounded velocities.

An estimate analogous to (7.18) gives that the last term in (7.21) can be estimated by $\|f_e^{j_1} - f_e^j\|_{r-1,T'}$ multiplied by a number proportional to $T' = \tilde{T}$ which can be chosen suitably small. It implies that $\{f_e^j\}$ is Cauchy in the $\| \ \|_{r-1,\tilde{T}}$ norm, for all $t \leqslant \tilde{T}$ the sequence $\{f_e^j(t)\}$ is Cauchy in the $\| \ \|_r$ norm, and $f = f_i + f_e$ is a solution of the Enskog equation in integral form with the boundary conditions. We remark that the length of the interval is dependent on $r$. The solution on the large interval $[0, T_1]$ is constructed by induction using the apriori estimates (2.21)–(2.24) and Proposition 7.6.

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## REFERENCES

1. L. Arkeryd, On the Enskog equation in two space variables, *Transport Theory and Stat. Phys.* **15**:673–691 (1986).
2. L. Arkeryd, Existence theorems for certain kinetic equations and large data, *Arch. Rat. Mech. Anal.* **103**:139–149 (1988).
3. L. Arkeryd, On the Enskog equation with large initial data, *SIAM J. Math. Anal.* **21**:631–646 (1990).
4. L. Arkeryd and C. Cercignani, Global existence in $L^1$ for the Enskog equation and convergence of the solutions to solutions of the Boltzmann equation, *J. of Stat. Phys.* **59**:845–867 (1990).
5. L. Arkeryd and N. B. Maslova, Boundary value problems and diffuse reflection, *J. Stat. Phys.* **77**:1051–1077 (1994).
6. L. Arkeryd and A. Nouri, Asymptotics of the Boltzmann equation with diffuse reflection boundary conditions, to appear in *Monatsheft für Mathematik* (Nice, 1994).
7. T. Carleman, *Théorie cinétique des gas* (Almqvist & Wiksell, Uppsala, 1957).
8. C. Cercignani, *The Boltzmann equation and its applications* (Springer, New York, 1987).
9. R. J. DiPerna and P. L. Lions, On the Cauchy problem for Boltzmann equation: Global existence and weak stability, *Ann. Math.* **130**:321–366 (1989).
10. D. Enskog, Kinetische Theorie der Wärmeleitung, Reibund und Selbstdiffusion in gevissen verdichteten Gasen und Flüssigkeiten, *Kungl. Svenska Vetenskapsakad Handl.* **63**:3–44 (1922).
11. H. Federer, Colloqium in geometric measure theory, *Bull. Am. Math. Soc.* **84**:291–338 (1978).
12. A. Heintz, Boundary value problems for nonlinear Boltzmann equation in domains with irregular boundaries, Ph.D.Thesis (1986), Leningrad State University.

13. A. Heintz, On solutions of boundary value problems for the Boltzmann equation in domains with irregular boundaries, in *Statistical Mechanics, Numerical Method in Kinetic Theory of Gases, Novosibirsk*, 1986, pp. 148–154.
14. A. Heintz, Initial boundary value problems in irregular domains for nonlinear kinetic equations of Boltzmann type. Preprint Chalmers University of Technology. NO 1996-20/ISSN 0347-2809, Göteborg. 1996.
15. P. Resibois, *H*-theorem for the (modified) nonlinear Enskog equation, *J. of Stat. Phys.* **19**:593–609 (1978).
16. G. Toscani, On the Cauchy problem for the discrete Boltzmann equation with initial values in $L_1^+(R)$, Tech. Report, (Universita di Ferrara, 1987).
17. S. Ukai, Solutions of the Boltzmann equation, in *Patterns and Waves-Qualitative Analysis of Nonlinear Differential equations*, H. Fujita, J. L. Lions, Papanicolau, and H. B. Keller, eds., Vol. 18, (Kinokuniya, 1986, pp. 37–96).